COULOMB SYSTEMS CONFINED BY IDEAL DIELECTRIC BOUNDARIES

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Abstract

We consider two-dimensional Coulomb systems confined in a disk with ideal dielectric boundaries, in particular the two-component plasma when the coupling constant $\Gamma = 2$. Under these conditions the system is equivalent to a free fermion field, the grand-partition function can be written as a Pfaffian and therefore the model is exactly solvable. We obtain analytic expressions for the grand potential, densities and correlations. We confirm that the grand potential exhibits a universal finite-size correction predicted in previous works.

Resumen

Estudiamos sistemas de Coulomb bidimensionales confinados en un disco por dieléctricos ideales (condiciones de frontera de Neumann para el potencial eléctrico). En particular estudiamos el plasma de dos componentes cuando la constante de acoplamiento $\Gamma = 2$. En estas condiciones el sistema es equivalente a una teoría de fermiones libres, la función de gran partición se puede escribir como un Pfaffiano y así se puede resolver exactamente el modelo. Obtenemos expresiones analíticas para el gran potencial, las densidades y correlaciones de carga en el sistema. Confirmamos además que el gran potencial tiene una corrección de talla finita universal que habia sido predicha anteriormente.

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1. Introduction. This paper is devoted to study a special class of two-dimensional solvable models of classical Coulomb systems. Beside their intrinsic interest as solvable models, these systems are also very useful to give hints on the properties of more complex systems of charged particles (electrolytes, plasmas,...). Here we give a short report on the study the two-dimensional two-component plasma (TCP) confined in a disk of radius R with Neumann boundary conditions imposed to the electric potential. This situation is obtained if the exterior of the disk is made of a dielectric with vanishing dielectric constant (an ideal dielectric). Further details can be found in Ref. [1].

The TCP is composed of two species of point particles with charges $\pm q$. The coupling constant is $\Gamma = \beta q^2 = q^2/k_B T$. In two-dimensions the Coulomb potential is logarithmic. For the present case with Neumann boundary conditions the Coulomb potential reads $v(\mathbf{r}, \mathbf{r}') = -\ln |z - z'||R^2 - z\bar{z}'|/a^2 R$ with z the complex coordinate of point $\mathbf{r}$ and a is an irrelevant length scale.

Some time ago [2, 3, 4] it was shown that a Coulomb system confined in a domain of size R and Euler characteristic χ with conformally invariant boundary conditions for the electric potential should exhibit a logarithmic universal finite-size correction of the form $(\chi/6)\ln R$ in the grand potential and in the free energy. Neumann boundary conditions being conformally invariant it is expected in our case to find a correction $(1/6)\ln R$.

The TCP was solved two decades ago by Gaudin [5] and since then much work has been done on that model [3, 4, 6, 7]. In particular it was solved with several types of boundary conditions. However it was not until very recently that the model could be solved for a system near a plane wall made of an ideal dielectric [8]. The main difficulty to solve the TCP for this special case of boundary conditions is that the system must be mapped onto a four-component free fermion field instead of a two-component free fermion field as in all other cases of boundaries conditions.

In the next section we show how the system can be mapped onto a free fermion field theory and therefore one can compute the grand potential. In Sec. 3 we show the results for the grand potential and the finite-size expansions.

2. Mapping onto a four-component free Fermi field. Starting first with a lattice model with sites $u_i (v_i)$ for positive (negative) particles and position dependent fugacities ζ the grand partition function is

$$\Xi = 1 + \sum_{N=1}^{\infty} \sum_{u_i, v_i} \prod_{i=1}^{N} \zeta(u_i)\zeta(v_i) e^{-\beta H_N} \tag{1}$$

When $\Gamma = 2$ the Boltzmann factor can be written as

$$e^{-\beta H_N} = (-1)^N \det \begin{pmatrix} \frac{a}{u_i - v_j} & \frac{aR}{u_i \bar{v}_j - R^2} \\ \frac{aR}{R^2 - \bar{u}_i v_j} & \frac{a}{\bar{v}_j - \bar{u}_i} \end{pmatrix} \tag{2}$$

It can be shown (see Refs. [1, 8]) that the grand partition function can be written as a ratio of two Pfaffians $\Xi = \mathrm{Pf}(A + M)/\,\mathrm{Pf}(A)$ with the antisymmetric matrices A and M defined by

$$(A^{-1})_{uv} = \begin{pmatrix} \frac{a}{u-v} & \frac{aR}{u\bar{v} - R^2} \\ \frac{aR}{R^2 - \bar{u}v} & \frac{a}{\bar{v} - \bar{u}} \end{pmatrix} \qquad M_{uu'} = \delta_{uu'} \begin{pmatrix} 0 & \zeta(u) \\ -\zeta(u) & 0 \end{pmatrix} \tag{3}$$

and $M_{uv} = 0, (A^{-1})_{uu'} = (A^{-1})_{vv'} = 0$. This is done introducing a set of four anticommuting Grassmann variables for each site of the lattice. Then, the grand partition function becomes $\Xi = (\det(1 + K))^{1/2}$ with $K = MA^{-1}$. In the continuum limit where the spacing the lattice goes to zero one can recognize in K the inverse of the Dirac operator

$$(K^{-1})_{ss'} = 2m\delta_{-ss'} \begin{pmatrix} 0 & \partial_{\bar{z}} \\ \partial_z & 0 \end{pmatrix} \quad \text{acting on a four component spinor} \quad \begin{pmatrix} \psi_-^{(1)} \\ \psi_-^{(1)} \\ \psi_+^{(2)} \\ \psi_+^{(2)} \end{pmatrix} \tag{4}$$

s and s' label the sign of the particles and m is a rescaled fugacity. To compute the grand potential one must find the eigenvalues of K or equivalently the eigenvalues of the Dirac operator with the boundary conditions $\psi_s^{(1)}(\mathbf{R}) + e^{i\phi}\psi_s^{(2)}(\mathbf{R}) = 0$ for a point $z = Re^{i\phi}$ on the boundary of the disk.

3. Results. Solving the eigenvalue problem for K one finds the grand potential in terms of the modified Bessel functions I_ℓ

$$\beta\Omega = -\sum_{\ell=0}^{\infty} \ln \left[\left(\frac{2}{mR}\right)^{2\ell} (\ell!)^2 (I_\ell(mR)^2 - I_{\ell+1}(mR)^2) \right] \tag{5}$$

The large-R expansion of the grand potential is found to be

$$\beta\Omega = -\beta p \pi R^2 + \beta\gamma 2\pi R + \frac{1}{6}\ln mR + O(1) \tag{6}$$

with the bulk pressure $\beta p_b = m^2(\ln(2/m\sigma)+1)$ and the surface tension $\beta\gamma = m/4$. The diameter of the particles σ is the cutoff of the theory: in the continuum limit $\sigma \to 0$ the bulk pressure is divergent due to the collapse of particles of opposite sign [6]. The system shows indeed the expected logarithmic finite-size correction $(1/6)\ln R$.

4. Conclusion. We showed how the classical TCP in a disk with ideal dielectric boundary conditions can be mapped onto a quantum free fermionic field theory and we solved it. The grand potential was calculated and it exhibits universal finite-size corrections predicted in previous works [2, 3, 4]. By using the same method, computing the propagator of the equivalent fermionic field theory, one can also compute the density and correlations functions of the classical Coulomb system. The reader can consult Refs. [1, 8] for further details on these calculations and the behavior of the correlation functions. This work is supported by COLCIENCIAS and BID under project # 1204-05-10078.

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